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[Fall 2025] ECE 5290/7290 and ORIE 5290
Distributed Optimization for Machine Learning and AI

Homework 2

Gradescope Due: September 28th at 5PM

Objective of This Assignment

The objective of this assignment is to deepen your understanding of fundamental gradient-based optimization algorithms and their convergence properties. You will analyze the behavior of these methods on a variety of problem classes, including ill-conditioned quadratic, constrained, and nonconvex functions. The problems will guide you through advanced, practical topics such as the acceleration provided by momentum (the Heavy-ball method), and the variance and trade-offs of Stochastic Gradient Descent (SGD).

Instruction of Homework Submission

This assignment includes both an analytical part and a coding part (Problem 5b). We will use Gradescope to check the correctness of your code. Therefore, you will see two separate assignments on Gradescope. For the analytical part, please follow the same instructions as Homework 1 and upload your work to **Homework 2**. For the coding part, please read the instructions below.

(a) A starter .py file has been provided for Problem 5b. Do not change the function names, the parameters of these functions, or the filename.

(b) When you are done, upload your completed .py file to **Homework 2 Coding** on Gradescope.

Question 1: The Impact of Condition Number on GD (20 points)

This problem explores the practical consequences of the condition number on the convergence of Gradient Descent (GD). Consider the quadratic objective $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x}$.

(a) **(5 points)** Consider two matrices:

$$\mathbf{Q}_1 = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Q}_2 = \begin{pmatrix} 5.5 & 4.5 \\ 4.5 & 5.5 \end{pmatrix}$$

Calculate the condition number $\kappa = \lambda_1/\lambda_n$ for both $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Which one is better-conditioned?

(b) **(5 points)** For both quadratic objectives defined by $\mathbf{Q}_1$ and $\mathbf{Q}_2$, write down the linear convergence rate for GD using the optimal constant stepsize $\eta = \frac{2}{\lambda_1+\lambda_n}$, in terms of the contraction factor $\rho = \frac{\kappa-1}{\kappa+1}$.

(c) **(10 points)** Sketch the contour plots for both functions. On each plot, sketch the expected path of Gradient Descent starting from a point like $\mathbf{x}^0 = [1, 1]^\top$. Briefly explain how the different condition numbers lead to different convergence behaviors.

Question 2: Stochastic Gradient Descent (25 points)

Consider the problem for linear regression: $F(\boldsymbol{\theta}) = \frac{1}{N}\sum_{i=1}^{N} f_i(\boldsymbol{\theta})$, where $f_i(\boldsymbol{\theta}) = \frac{1}{2}(\mathbf{a}_i^\top \boldsymbol{\theta} - y_i)^2$. The stochastic gradient for a single sample i_t (chosen uniformly at random) is $g_{i_t}(\boldsymbol{\theta}) = \nabla f_{i_t}(\boldsymbol{\theta})$.

(a) **(5 points) Unbiased Estimator:** Show that the stochastic gradient is an unbiased estimator of the true gradient, i.e., $\mathbb{E}_{i_t}[g_{i_t}(\boldsymbol{\theta})] = \nabla F(\boldsymbol{\theta})$.

(b) **(10 points) Variance of the Stochastic Gradient:** Derive an expression for the variance of the stochastic gradient estimator, defined as $\sigma^2_{\boldsymbol{\theta}} = \mathbb{E}_{i_t}[||g_{i_t}(\boldsymbol{\theta}) - \nabla F(\boldsymbol{\theta})||_2^2]$.

(Hint: You can use the identity $\mathbb{E}[||\mathbf{X} - \mathbb{E}[\mathbf{X}]||^2] = \mathbb{E}[||\mathbf{X}||^2] - ||\mathbb{E}[\mathbf{X}]||^2$. Your final expression will be in terms of the gradients of the individual component functions, $\nabla f_i(\boldsymbol{\theta})$).

(c) **(10 points) The Power of Mini-batching - For ECE/ORIE 5290 students:** Consider a mini-batch stochastic gradient as

$$g_{\mathcal{B}}(\boldsymbol{\theta}) = \frac{1}{B}\sum_{j \in \mathcal{B}} g_j(\boldsymbol{\theta}),$$

where $\mathcal{B}$ is a mini-batch of size B sampled uniformly *with replacement* from $\{1, ..., N\}$. [1]

i) Show that this mini-batch estimator is also unbiased.

ii) Show that the variance of the mini-batch estimator is reduced by a factor of B:

$$\mathrm{Var}(g_{\mathcal{B}}(\boldsymbol{\theta})) = \mathbb{E}_{\mathcal{B}}[||g_{\mathcal{B}}(\boldsymbol{\theta}) - \nabla F(\boldsymbol{\theta})||_2^2] = \frac{1}{B}\sigma^2_{\boldsymbol{\theta}}$$

(d) **(10 points) Connection to convergence - For ECE 7290 students:** Look at the long-term error term in the SGD convergence from the lecture: $\frac{\eta L \sigma_g^2}{2\mu}$. Based on your result from part (c), if we use a mini-batch of size B, how does this error term change? What does this tell you about the trade-off between computational cost per iteration and convergence behavior when choosing a batch size?

Question 3: GD on Nonconvex Functions (25 points)

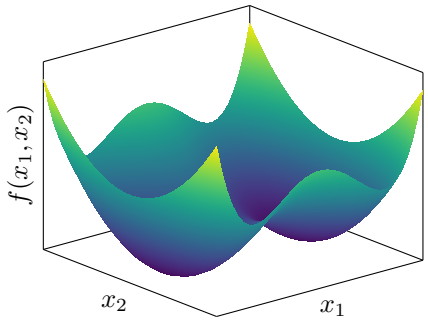
Consider the 2D nonconvex function below, which is a classic example of a surface with multiple minima and a saddle point. A plot of the function is provided for visualization.

$$f(x_1, x_2) = x_1^4 - 2x_1^2 + x_2^2$$

(a) **(5 points)** Find all stationary points of f by setting its gradient $\nabla f(x_1, x_2)$ to zero.

(b) **(5 points)** Compute the Hessian matrix $\nabla^2 f(x_1, x_2)$.

[1] Sampling *with replacement* means that after a data point is selected for the mini-batch, it is "put back" into the dataset and is eligible to be selected again for the same mini-batch. This is a common assumption that simplifies the mathematical analysis of variance, as it ensures every selection is an independent event.



(c) **(5 points)** For each stationary point you found, evaluate the Hessian and use its eigenvalues to classify the point as a local minimum, local maximum, or a saddle point.

(d) **(10 points) For ECE/ORIE 5290 students:** One of the stationary points is a saddle point. If you were to run GD and initialize it *exactly* at this saddle point, what would happen? Now, what would happen if you initialized it *very close* to the saddle point (e.g., with a tiny random perturbation)? Briefly explain your reasoning.

(d) **(10 points) For ECE 7290 students:** The stationary point at $(0, 0)$ is a saddle point. Let's analyze how Gradient Descent escapes it.

(1) Consider an initial point very close to the saddle, $\mathbf{x}^0 = [\epsilon, \epsilon]^\top$, for a very small $\epsilon > 0$. Compute the gradient $\nabla f(\mathbf{x}^0)$. For a very small ϵ, you can ignore higher-order terms (like ϵ^3). What is the approximate direction of the first GD step, $-\nabla f(\mathbf{x}^0)$?

(2) The Hessian at the saddle point, $\nabla^2 f(0, 0)$, has one negative eigenvalue. The corresponding eigenvector points in the direction of negative curvature—the direction the algorithm will "roll off" the saddle. What is this eigenvector?

(3) Compare your answers from (1) and (2). What do you notice about the relationship between the GD step direction and the eigenvector corresponding to the negative eigenvalue? Briefly explain why this ensures GD can escape saddle points in practice.

Question 4: Coding - Implementing and Comparing Optimizers (30 points)

In this problem, you will implement and compare different first-order optimization algorithms on a logistic regression problem. The dataset is specifically designed to be **ill-conditioned** to highlight the performance differences between the algorithms.

(a) Data Generation and Visualization (5 points)

Use the provided Python code snippet to generate and visualize the 2D synthetic dataset. The features have been intentionally scaled to create an ill-conditioned problem (notice the different scales on the x and y axes). Explain why this feature scaling leads to an ill-conditioned Hessian for the logistic regression loss.

```
import numpy as np
from sklearn.datasets import make_classification
import matplotlib.pyplot as plt
```

```
# Generate a synthetic dataset
X, y = make_classification(n_samples=200, n_features=2, n_redundant=0,
                           n_informative=2, random_state=1, n_clusters_per_class=1)

# Induce ill-conditioning by scaling one feature
X[:, 1] = X[:, 1] * 20

# Plotting the data
plt.scatter(X[:, 0], X[:, 1], c=y, edgecolors='k', cmap=plt.cm.Paired)
plt.xlabel("Feature 1")
plt.ylabel("Feature 2 (scaled)")
plt.title("Ill-Conditioned Synthetic Dataset")
plt.show()
```

(b) Implementation (10 points)

Implement the following three algorithms to minimize the Binary Cross-Entropy loss (introduced in the class) on this dataset. You will also need to implement the sigmoid function and the BCE loss and its gradient.

(a) Batch Gradient Descent (GD)

(b) Heavy-ball Method (GD with Momentum)

(c) Stochastic Gradient Descent (SGD) with a mini-batch size of your choice (e.g., $B = 8$).

(c) Hyperparameter Tuning (5 points)

For each of the three algorithms, experiment to find a "good" set of hyperparameters (learning rate η and, for momentum methods, the momentum parameter β). There is no single "correct" answer, but you should demonstrate that you've tried a few values to get reasonable performance. Report the final hyperparameters you chose for each algorithm.

(d) Comparison and Analysis (10 points)

Using your best hyperparameters from part (c), run all four optimizers from the same initial point $\boldsymbol{\theta}^0 = \mathbf{0}$ for a fixed number of epochs (e.g., 50 epochs).

(a) **Plot 1:** Generate a single plot showing the **Loss vs. Epochs** for all four methods. The y-axis should be on a logarithmic scale.

(b) **Plot 2:** Generate a single plot showing the 2D data points and the final **decision boundary** learned by each of the four algorithms.

(c) **Written Analysis:** In a short paragraph, answer the following:

- Why does the path of standard GD likely show slow, zig-zagging behavior on this dataset?
- How does the Heavy-ball method improve upon this? What is the key difference you observe in their loss curves?
- Describe the SGD loss curve. Why is it noisy, and what is its main advantage in the early epochs compared to the batch methods?